The hidden algebraic structure of the Kaup–Broer hierarchy

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Abstract. In this paper, we investigate the conformal algebra associated with the coupled Kaup–Broer hierarchy which is obtained from the coupled AKNS hierarchy by gauge transformation. We find that the algebraic structure hidden behind the coupled Kaup–Broer hierarchy can be revealed after mapping the Lax operator of the coupled Kaup–Broer hierarchy to a Yajima–Oikawa-type Lax operator. Furthermore, we show that the second Hamiltonian structure of the coupled Kaup–Broer hierarchy can be simplified by factorizing the Lax operator in multiplicative form and thus obtain the free-field realization of the associated algebras.

1. Introduction

In the past few years, there have been several intensive studies on the relationships between conformal field theory and integrable systems which include, in particular, exploration of the role played by the classical W-algebras in integrable systems (see [1] and references therein). A famous example is the W_n algebra constructed from the second Gelfand–Dickey (GD) structure of the nth Korteweg–de Vries (KdV) hierarchy [2, 3]. By factorizing the KdV-Lax operator, the second Hamiltonian structure is transformed into a much simpler one. Therefore, the factorization not only provides a Miura transformation which maps the nth KdV hierarchy to the corresponding modified hierarchy, but it also gives a free-field realization of the W_n-algebra. This is what we called the Kupershmidt–Wilson (KW) theorem [4, 5]. In general, the above scheme is encoded in the particular form of the Lax operator and its associated Poisson structure. Therefore, the number of integrable hierarchies where the KW theorem works is quite limited.

Recently, the so-called constrained Kadomtsev–Petviashvili (cKP) hierarchy has been proposed and studied [6–10] (see, e.g., [11] for a review). The cKP hierarchy is the Nth KP hierarchy restricted to a pseudo-differential operator of the form

$$L_{(N,M)} = \partial^N + u_2 \partial^{N-2} + \cdots + u_0 + \sum_{i=1}^{M} \phi_i \partial^{-1} \psi_i \tag{1.1}$$

which satisfies the hierarchy equations

$$\partial_k L_{(N,M)} = [(L_{(N,M)})_+^{k/N}, L_{(N,M)}]. \tag{1.2}$$

The cKP hierarchy contains many well known integrable systems, for example, the AKNS hierarchy ($N = 1, M = 1$), the Yajima–Oikawa (YO) hierarchy ($N = 2, M = 1$) [12] and the Melnikov hierarchy ($N = 3, M = 1$) [13]. For $M > 1$, they represent the hierarchies of the fields u_i coupled with the eigenfunctions ϕ_i and adjoint eigenfunctions ψ_i.

Apart from the cKP hierarchy, the constrained modified KP (cmKP) hierarchy [10] can also be defined from the Lax operator (1.1) using the gauge transformation [14]

$$K_{(N,M)} = \phi_1^{-1} L_{(N,M)} \phi_1 \tag{1.3}$$

$$= \partial^N + v_1 \partial^{N-1} + \cdots + \partial^{-1} v_{N+1} + \sum_{i=1}^{M-1} q_i \partial^{-1} r_i. \tag{1.4}$$

The hierarchy equations then become

$$\partial_k K_{(N,M)} = [(K_{(N,M)})_{>0}^{k/N}, K_{(N,M)}] \tag{1.5}$$

which contains the Kaup–Broer (KB) hierarchy ($N = 1$, $M = 1$) [15, 16] and the extended modified KdV hierarchy ($N = 2$, $M = 1$).

The bi-Hamiltonian structures of the cKP and cmKP hierarchies have been obtained in [10, 17]. The second Hamiltonian structures are somewhat different from the standard GD structure associated with the Nth KdV hierarchy. Therefore, it is very interesting to investigate the algebraic structures associated with their second structures defined by (1.1) and (1.4). Besides the Nth KdV hierarchy, the cases for $L_{(N,M)}$ with $N \geqslant 2$, $M = 1$ have been investigated [18, 19]. However, to the best of our knowledge, the analysis for the cases with $N = 1$, $M > 1$ is still lacking. It is the purpose of this paper to investigate this case and to fill the gap in this approach.

This paper is organized as follows. In section 2 we first consider the coupled AKNS hierarchy. We calculate its Poisson brackets from its second Hamiltonian structure and discuss its associated conformal algebra. Then in section 3 we perform a gauge transformation to obtain the coupled Kaup–Broer (cKB) hierarchy and the corresponding Poisson brackets. We find that after mapping the Lax operator to a YO-type Lax operator, the Poisson structure becomes the sum of the second and the third GD brackets defined by the YO-type Lax operator. Therefore, the Poisson algebra associated with the cKB hierarchy is encoded in the covariantization of the YO-type Lax operator. In section 4 we simplify this Poisson structure by factorizing the Lax operator in multiplicative form and thus obtain the free-field realization of the associated algebra. Conclusions and discussions are presented in section 5.

2. Coupled AKNS hierarchy

The general coupled AKNS hierarchy is the ordinary KP hierarchy restricted to the pseudo-differential operator of the form

$$L_{(1,M)} = \partial + \sum_{i=1}^{M} \phi_i \partial^{-1} \psi_i. \tag{2.1}$$

The evolution of the system is given by

$$\partial_k L_{(1,M)} = [(L_{(1,M)}^k)_+, L_{(1,M)}] \tag{2.2}$$

$$\partial_k \phi_i = ((L_{(1,M)}^k)_+ \phi_i)_0 \qquad \partial_k \psi_i = -((L_{(1,M)}^k)_+^* \psi_i)_0 \tag{2.3}$$

where ϕ_i and ψ_i are eigenfunctions and adjoint eigenfunctions, respectively. $((A)_\pm$ denote the differential part and the integral part of the pseudo-differential operator A, respectively, $(A)_0$ denotes the zeroth-order term, and * stands for the conjugate operation: $(AB)^* = B^* A^*$, $\partial^* = -\partial$, $f(x)^* = f(x)$.) It can be proved that (2.2) is consistent with (2.3).

The second Hamiltonian structure associated with $L_{(1,M)}$ is given by [10]

$$\Theta_2^{AKNS}\left(\frac{\delta H}{\delta L_{(1,M)}}\right) = \left(L_{(1,M)}\frac{\delta H}{\delta L_{(1,M)}}\right)_+ L_{(1,M)} - L_{(1,M)}\left(\frac{\delta H}{\delta L_{(1,M)}}L_{(1,M)}\right)_+$$
$$+\left[L_{(1,M)}, \int^x \mathrm{res}\left[L_{(1,M)}, \frac{\delta H}{\delta L_{(1,M)}}\right]\right] \tag{2.4}$$

where the last term in (2.4) is just the Dirac constraint imposed by $u_1 = 0$ on $L_{(1,M)}$. In the following, without loss of generality, let us discuss the $M = 2$ case. The coupled AKNS hierarchy is defined by

$$L_{(1,2)} = \partial + \phi_1\partial^{-1}\psi_1 + \phi_2\partial^{-1}\psi_2. \tag{2.5}$$

The first non-trivial equations are given by

$$\partial_2\phi_1 = \phi_{1xx} + 2\phi_1^2\psi_1 + 2\phi_1\phi_2\psi_2 \tag{2.6}$$
$$\partial_2\phi_2 = \phi_{2xx} + 2\phi_2^2\psi_2 + 2\phi_1\phi_2\psi_1 \tag{2.7}$$
$$\partial_2\psi_1 = -\psi_{1xx} - 2\phi_1\psi_1^2 - 2\phi_2\psi_1\psi_2 \tag{2.8}$$
$$\partial_2\psi_2 = -\psi_{2xx} - 2\phi_2\psi_2^2 - 2\phi_1\psi_1\psi_2 \tag{2.9}$$

and

$$\partial_3\phi_1 = \phi_{1xxx} + 3(\phi_1^2)_x\psi_1 + 6(\phi_1\phi_2)_x\psi_2 \tag{2.10}$$
$$\partial_3\phi_2 = \phi_{2xxx} + 3(\phi_2^2)_x\psi_2 + 6(\phi_1\phi_2)_x\psi_1 \tag{2.11}$$
$$\partial_3\psi_1 = \psi_{1xxx} + 3\phi_1(\psi_1^2)_x + 3\phi_2(\psi_1\psi_2)_x \tag{2.12}$$
$$\partial_3\psi_2 = \psi_{2xxx} + 3\phi_1(\psi_1\psi_2)_x + 3\phi_2(\psi_2^2)_x. \tag{2.13}$$

Of course, as we drop the variables ϕ_2 and ψ_2, the first AKNS equations are recovered.

From the second Hamiltonian formulation of these equations, (2.4) gives the basic second Poisson brackets as follows

$$\{\phi_i(x), \phi_j(y)\} = -\phi_i(x)\epsilon(x - y)\phi_j(y) - \phi_j(x)\epsilon(x - y)\phi_i(y)$$
$$\{\psi_i(x), \psi_j(y)\} = -\psi_i(x)\epsilon(x - y)\psi_j(y) - \psi_j(x)\epsilon(x - y)\psi_i(y)$$
$$\{\phi_i(x), \psi_j(y)\} = \delta_{ij}L_{(1,2)}\delta(x - y) + \phi_i(x)\epsilon(x - y)\psi_j(y) \tag{2.14}$$

where $\epsilon(x - y) \equiv \partial_x^{-1}\delta(x - y)$ is the antisymmetric step function. We note that equations (2.14) are also obtained by factorizing the recursion operator [20].

To explore the conformal algebra associated with the Poisson brackets in (2.14), we observe that there is no natural choice for the Virasoro generator with dimension two from the Lax operator due to the fact that $N = 1$. However, if we set $t \equiv \phi_1\psi_1 + \phi_2\psi_2$, then the algebraic structure of the Poisson brackets is clear, i.e.

$$\{t(x), t(y)\} = [2t(x)\partial_x + t'(x)]\delta(x - y)$$
$$\{\phi_i(x), t(y)\} = [\phi_i(x)\partial_x + \phi_i'(x)]\delta(x - y)$$
$$\{\psi_i(x), t(y)\} = [\psi_i(x)\partial_x + \psi_i'(x)]\delta(x - y) \tag{2.15}$$

where the first bracket is just the Virasoro algebra without the central term, and ϕ_i and ψ_i are spin-1 fields with respect to the Virasoro generator t. Hence (2.15) forms a non-local extension of the Virasoro algebra by four spin-1 fields. We would like to remark that the algebra (2.15) can be generalized to the $M > 2$ case just by setting the Virasoro generator $t = \sum_{i=1}^M \phi_i\psi_i$.

3. Coupled Kaup–Broer hierarchy

In view of (1.3), the Lax operator of the cKB hierarchy is obtained by performing a gauge transformation on $L_{(1,2)}$ as follows

$$K_{(1,2)} = \phi_1^{-1} L_{(1,2)} \phi_1 \tag{3.1}$$

$$= \partial + v_1 + \partial^{-1} v_2 + q \partial^{-1} r \tag{3.2}$$

where

$$v_1 = \phi_1'/\phi_1 \qquad v_2 = \phi_1 \psi_1 \tag{3.3}$$

$$q = \phi_1^{-1} \phi_2 \qquad r = \phi_1 \psi_2. \tag{3.4}$$

The transformed Lax operator $K_{(1,2)}$ then satisfies the hierarchy equations

$$\partial_n K_{(1,2)} = [(K_{(1,2)}^n)_{\geqslant 1}, K_{(1,2)}]$$

$$\partial_n q = ((K_{(1,2)}^n)_{\geqslant 1} q)_0$$

$$\partial_n v_2 = -((K_{(1,2)}^n)_{\geqslant 1}^* v_2)_0 \qquad \partial_n r = -((K_{(1,2)}^n)_{\geqslant 1}^* r)_0. \tag{3.5}$$

The first non-trivial equations are given by

$$\partial_2 v_1 = (v_{1x} + 2v_2 + v_1^2 + 2qr)_x \tag{3.6}$$

$$\partial_2 v_2 = -(v_{2x} - 2v_1 v_2)_x \tag{3.7}$$

$$\partial_2 q = q_{xx} + 2v_1 q_x \tag{3.8}$$

$$\partial_2 r = -(r_x - 2v_1 r)_x \tag{3.9}$$

and

$$\partial_3 v_1 = (v_{1xx} + v_1^3 + v_1 v_{1x} + 6v_1 v_2 + 3q_x r + 6v_1 qr)_x \tag{3.10}$$

$$\partial_3 v_2 = (v_{2xx} - 3v_1 v_{2x} + 3v_1^2 v_2 + 3v_2^2 + 3qrv_2)_x \tag{3.11}$$

$$\partial_3 q = q_{xxx} + 3v_1 q_{xx} + 3(v_{1x} + v_1^2 + v_2 + qr)q_x \tag{3.12}$$

$$\partial_3 r = (r_{xx} - 3v_1 r_x + 3v_1^2 r + 3v_2 r + 3qr^2)_x \tag{3.13}$$

which reduce to the first KB equations by setting $q = r = 0$. On the other hand, the transformed second Hamiltonian structure now becomes [10, 17]

$$\Theta_2^{\text{cKB}} \left(\frac{\delta H}{\delta K_{(1,2)}} \right) = \left(K_{(1,2)} \frac{\delta H}{\delta K_{(1,2)}} \right)_+ K_{(1,2)} - K_{(1,2)} \left(\frac{\delta H}{\delta K_{(1,2)}} K_{(1,2)} \right)_+$$

$$+ \left[K_{(1,2)}, \left(K_{(1,2)} \frac{\delta H}{\delta K_{(1,2)}} \right)_0 \right] + \partial^{-1} \left(\text{res} \left[K_{(1,2)}, \frac{\delta H}{\delta K_{(1,2)}} \right] \right) K_{(1,2)}$$

$$+ \left[K_{(1,2)}, \int^x \text{res} \left[K_{(1,2)}, \frac{\delta H}{\delta K_{(1,2)}} \right] \right] \tag{3.14}$$

which leads to the basic second Poisson brackets as follows

$$\{v_1(x), v_1(y)\} = 2\partial_x \delta(x - y)$$

$$\{v_1(x), v_2(y)\} = [\partial_x^2 + v_1(x)\partial_x + v_1'(x) + q(x)r(x)]\delta(x - y) + q'(x)\epsilon(x - y)r(y)$$

$$\{v_1(x), q(y)\} = -q'(x)\epsilon(x - y)$$

$$\{v_1(x), r(y)\} = -r(x)\delta(x - y)$$

$$\{v_2(x), v_2(y)\} = [2v_2(x)\partial_x + v_2'(x)]\delta(x - y) + v_2(x)q(x)\epsilon(x - y)r(y)$$

$$\qquad\qquad + r(x)\epsilon(x - y)q(y)v_2(y)$$

$$\{v_2(x), q(y)\} = [-q(x)\partial_x - q'(x) + v_1(x)q(x)]\delta(x - y) - v_2(x)q(x)\epsilon(x - y)$$

$$-r(x)\epsilon(x-y)q^2(y)$$

$$\{v_2(x), r(y)\} = [r(x)\partial_x + r'(x) - v_1(x)r(x)]\delta(x-y) + r(x)\epsilon(x-y)q(y)r(y)$$

$$-r(x)\epsilon(x-y)v_2(y)$$

$$\{q(x), q(y)\} = -2q(x)\epsilon(x-y)q(y) + \epsilon(x-y)q^2(y) + q^2(x)\epsilon(x-y)$$

$$\{q(x), r(y)\} = [\partial_x + v_1(x)]\delta(x-y) + 2q(x)\epsilon(x-y)r(y) + \epsilon(x-y)v_2(y)$$

$$-\epsilon(x-y)q(y)r(y)$$

$$\{r(x), r(y)\} = -2r(x)\epsilon(x-y)r(y) \tag{3.15}$$

which are non-local as well. Now let us discuss the algebraic structure associated with the brackets (3.15). Based on the dimensional consideration, the Virasoro generator should be chosen as $t \equiv v_2 + v_1'/2 + qr$. Then from (3.15), we have

$$\{v_1(x), t(y)\} = [v_1(x)\partial_x + v_1'(x)]\delta(x-y)$$

$$\{t(x), t(y)\} = [\tfrac{1}{2}\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x-y)$$

$$\{q(x), t(y)\} = \tfrac{3}{2}q'(x)\delta(x-y) - \tfrac{1}{2}\epsilon(x-y)q''(y)$$

$$\{r(x), t(y)\} = [\tfrac{3}{2}r(x)\partial_x + r'(x)]\delta(x-y). \tag{3.16}$$

We see that v_1 and r are conformal spin-1 and spin-$\frac{3}{2}$ fields, respectively, but q is not a spin field. Therefore, we fail to covariantize the Lax operator $K_{(1,2)}$ directly. However, we note that if we take a derivative of the third bracket in (3.16), then q' becomes a spin-$\frac{3}{2}$ field, i.e.

$$\{q'(x), t(y)\} = [\tfrac{3}{2}q'(x)\partial_x + q''(x)]\delta(x-y). \tag{3.17}$$

This observation motivates us to consider the Lax operator $L_{(2,1)}$ defined by

$$L_{(2,1)} = \partial K_{(1,2)} \tag{3.18}$$

$$= \partial^2 + u_1\partial + u_2 + \phi\partial^{-1}\psi \tag{3.19}$$

where

$$u_1 = v_1 \qquad u_2 = v_2 + v_1' + qr$$

$$\phi = q' \qquad \psi = r. \tag{3.20}$$

Using (3.15) and (3.20), the Poisson brackets for $\{u_1, u_2, \phi, \psi\}$ now become simpler as

$$\{u_1(x), u_1(y)\} = 2\partial_x\delta(x-y)$$

$$\{u_1(x), u_2(y)\} = [-\partial_x^2 + u_1(x)\partial_x + u_1'(x)]\delta(x-y)$$

$$\{u_1(x), \phi(y)\} = \phi(x)\delta(x-y)$$

$$\{u_1(x), \psi(y)\} = -\psi(x)\delta(x-y)$$

$$\{\phi(x), \phi(y)\} = -2\phi(x)\epsilon(x-y)\phi(y) \tag{3.21}$$

etc. Note that these brackets are not the same as those constructed from the second GD brackets associated with the Lax operator (3.19), where the corresponding brackets are given by [10]

$$\{u_1(x), u_1(y)\} = -2\partial_x\delta(x-y)$$

$$\{u_1(x), u_2(y)\} = [\partial_x^2 - u_1(x)\partial_x - u_1'(x)]\delta(x-y)$$

$$\{u_1(x), \phi(y)\} = -\phi(x)\delta(x-y)$$

$$\{u_1(x), \psi(y)\} = \psi(x)\delta(x-y)$$

$$\{\phi(x), \phi(y)\} = -\phi(x)\epsilon(x-y)\phi(y). \tag{3.22}$$

Form (3.21), $L_{(2,1)}$ can be easily covariantized by setting the Virasoro generator $t \equiv u_2 - u_1'/2$, and

$$\{u_1(x), t(y)\} = [u_1(x)\partial_x + u_1'(x)]\delta(x - y)$$
$$\{t(x), t(y)\} = [\tfrac{1}{2}\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x - y)$$
$$\{\phi(x), t(y)\} = [\tfrac{3}{2}\phi(x)\partial_x + \phi'(x)]\delta(x - y)$$
$$\{\psi(x), t(y)\} = [\tfrac{3}{2}\psi(x)\partial_x + \psi'(x)]\delta(x - y). \tag{3.23}$$

Therefore, the Poisson algebra hidden behind $K_{(1,2)}$ is encoded in the conformal algebra associated with the operator $L_{(2,1)}$ and equation (3.23) shows that the algebra is a non-local extension of the Virasoro algebra by two spin-$\tfrac{3}{2}$ fields.

4. The free-field realization

We have discussed the Poisson algebra associated with the cKB hierarchy through the Lax operator $L_{(2,1)}$. In fact, we can go further to discuss the free-field realization of such an algebra. To do this, let us go back to the second Hamiltonian structure (3.14) and see why the simple expression (3.21) is possible. Using (3.18), it can be shown (see the appendix) that

$$\frac{\delta H}{\delta K_{(1,2)}}\partial^{-1} = \frac{\delta H}{\delta L_{(2,1)}} + O(\partial^{-3}). \tag{4.1}$$

Then the second structure defined by $K_{(1,2)}$ can be transformed to the one defined by $L_{(2,1)}$ as follows

$$\{F, G\} = \int \mathrm{res}\left(\frac{\delta F}{\delta K_{(1,2)}}\Theta_2^{\mathrm{cKB}}\left(\frac{\delta G}{\delta K_{(1,2)}}\right)\right) = \int \mathrm{res}\left(\frac{\delta F}{\delta L_{(2,1)}}\Omega\left(\frac{\delta G}{\delta L_{(2,1)}}\right)\right) \tag{4.2}$$

where

$$\begin{aligned}
\Omega\left(\frac{\delta G}{\delta L_{(2,1)}}\right) &= \partial\Theta_2^{\mathrm{cKB}}\left(\frac{\delta G}{\delta K_{(1,2)}}\right)\\
&= \left(L_{(2,1)}\frac{\delta G}{\delta L_{(2,1)}}\right)_+ L_{(2,1)} - L_{(2,1)}\left(\frac{\delta G}{\delta L_{(2,1)}}L_{(2,1)}\right)_+\\
&\quad + \left[L_{(2,1)}, \int^x \mathrm{res}\left[L_{(2,1)}, \frac{\delta G}{\delta L_{(2,1)}}\right]\right].
\end{aligned} \tag{4.3}$$

Besides the second GD structure, the last term in (4.3) is called the third GD structure which is compatible with the second one. Thus the Hamiltonian structure associated with $L_{(2,1)}$ is the sum of the second and the third GD structures.

Next, we want to show that the Poisson structure (4.3) has a very interesting property under factorization of the operator $L_{(2,1)}$ in a multiplicative form. Since the operator $L_{(2,1)}$ has multi-boson representations [21–25], one can factorize $L_{(2,1)}$ into the following form

$$L_{(2,1)} = (\partial - a_1)(\partial - a_2)(\partial - a_3)(\partial - b_1)^{-1} \tag{4.4}$$

where the variables $\{u_1, u_2, \phi, \psi\}$ and $\{a_1, a_2, a_3, b_1\}$ are related by

$$\begin{aligned}
u_1 &= b_1 - (a_1 + a_2 + a_3)\\
u_2 &= u_1 b_1 + 2b_1' + a_1 a_2 + a_2 a_3 + a_1 a_3 - a_2' - 2a_3'\\
\phi &= \exp\left[\int^x b_1\right](u_2 b_1 + u_1 b_1' + b_1'' - a_1 a_2 a_3 + a_1 a_3' + a_2' a_3 + a_2 a_3' - a_3'')\\
\psi &= \exp\left[-\int^x b_1\right]
\end{aligned} \tag{4.5}$$

which is called the Miura transformation. Now let us first consider the second GD bracket (the first two terms of (4.3)) under the factorization (4.4). Thanks to the generalized KW theorem [21–25], the second GD bracket can be simplified as

$$\{a_i(x), a_j(y)\}_2^{GD} = -\delta_{ij}\partial_x\delta(x-y)$$
$$\{b_1(x), b_1(y)\}_2^{GD} = \partial_x\delta(x-y)$$
$$\{a_i(x), b_1(y)\}_2^{GD} = 0. \tag{4.6}$$

Hence the remaining tasks are to study the third GD structure. In our previous work [26], we have shown that the third structure (the last term of (4.3)) also has a very nice property under factorization of the Lax operator containing inverse linear terms (4.4). It turns out that [26]

$$\{F, G\}_3^{GD} = \int \mathrm{res}\left(\frac{\delta F}{\delta L_{(2,1)}}\left[L_{(2,1)}, \int^x \mathrm{res}\left[L_{(2,1)}, \frac{\delta G}{\delta L_{(2,1)}}\right]\right]\right)$$
$$= \int \left(\sum_{i=1}^{3}\frac{\delta F}{\delta a_i} + \frac{\delta G}{\delta b_1}\right)\left(\sum_{j=1}^{3}\frac{\delta F}{\delta a_j} + \frac{\delta G}{\delta b_1}\right)' \tag{4.7}$$

which leads to

$$\{a_i(x), a_j(y)\}_3^{GD} = \{a_i(x), b_1(y)\}_3^{GD} = \{b_1(x), b_1(y)\}_3^{GD} = \partial_x\delta(x-y). \tag{4.8}$$

Combining (4.6) with (4.8) we obtain

$$\{a_i(x), a_j(y)\} = (1 - \delta_{ij})\partial_x\delta(x-y)$$
$$\{b_1(x), b_1(y)\} = 2\partial_x\delta(x-y)$$
$$\{a_i(x), b_1(y)\} = \partial_x\delta(x-y). \tag{4.9}$$

Therefore, the Lax operator $K_{(1,2)}$ has a simple and local realization of its Poisson structures. Furthermore, we can diagonalize the Poisson bracket matrix of (4.9) to obtain the free-field realization. After some simple algebra, the variables $\boldsymbol{\omega}^T = (\omega_1, \omega_2, \omega_3, \omega_4) \equiv (a_1, a_2, a_3, b_1)$ can be expressed as

$$\omega_i = \boldsymbol{h}_i^T \cdot \boldsymbol{e} \tag{4.10}$$

where

$$\boldsymbol{h}_1^T = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \qquad \boldsymbol{h}_2^T = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$
$$\boldsymbol{h}_3^T = \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \qquad \boldsymbol{h}_4^T = \left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \tag{4.11}$$

and $\boldsymbol{e}^T \equiv (e_1, e_2, e_3, e_4)$ which satisfy

$$\{e_i(x), e_j(y)\} = \lambda_i\delta_{ij}\partial_x\delta(x-y) \tag{4.12}$$

with $\lambda_1 = \lambda_2 = -1, \lambda_3 = (2 + \sqrt{3}), \lambda_4 = (2 - \sqrt{3})$. Thus the Lax operator $K_{(1,2)}$ of the cKB hierarchy can be expressed in terms of the free fields $\{e_i\}$ as

$$K_{(1,2)} = \partial^{-1}(\partial - \boldsymbol{h}_1^T \cdot e)(\partial - \boldsymbol{h}_2^T \cdot e)(\partial - \boldsymbol{h}_3^T \cdot e)(\partial - \boldsymbol{h}_4^T \cdot e)^{-1}. \tag{4.13}$$

5. Conclusions

We have investigated the hidden conformal algebras of the cKB hierarchy. We have shown that the second Hamiltonian structure of the cKB hierarchy has a very simple realization. In terms of $\{a_1, a_2, a_3, b_1\}$ the Lax operator $K_{(1,2)}$ can be factorized as

$$K_{(1,2)} = \partial^{-1}(\partial - a_1)(\partial - a_2)(\partial - a_3)(\partial - b_1)^{-1} \tag{5.1}$$

and the second Poisson brackets (3.15) mapped to a much simpler form (4.9). Thus the free-field realization of the associated algebra is obtained. In general, we should consider the multi-constraint KP hierarchy with the Lax operator of the form (2.1). We can follow the strategy in the appendix to prove without difficulty that the Hamiltonian structure associated with the operator $L_{(N+1,M-1)} \equiv \partial K_{(N,M)}$ is just the sum of the second and third GD structures (4.3). Thus by applying the previous result [26], the Lax operator of the cmKP hierarchy can be factorized as

$$K_{(N,M)} = \partial^{-1}(\partial - a_1)\cdots(\partial - a_n)(\partial - b_1)^{-1}\cdots(\partial - b_m)^{-1} \tag{5.2}$$

where $n = N + M$, $m = M - 1$ and the simplified Poisson brackets turn out to be

$$\{a_i(x), a_j(y)\} = (1 - \delta_{ij})\partial_x\delta(x - y)$$
$$\{b_i(x), b_j(y)\} = (1 + \delta_{ij})\partial_x\delta(x - y)$$
$$\{a_i(x), b_j(y)\} = \partial_x\delta(x - y). \tag{5.3}$$

We remark that equations (5.2) and (5.3) can be viewed as the generalization of the KW theorem for the Nth cKB hierarchy associated with the Lax operator $K_{(N,1)}$ [26, 27]. Finally, we would like to mention that the Poisson bracket matrix (5.3) is symmetric and non-singular. It is not difficult to diagonalize the matrix to obtain the free-field realization which would be useful to quantize the W-algebras associated with the cmKP hierarchy. We will leave these discussions to another publication.

Appendix. Proof of (4.1)

In this appendix we give a proof of (4.1). From (3.20) we have

$$\frac{\delta H}{\delta v_1} = \frac{\delta H}{\delta u_1} - \left(\frac{\delta H}{\delta u_2}\right)' \qquad \frac{\delta H}{\delta v_2} = \frac{\delta H}{\delta u_2} \tag{A.1}$$

$$\frac{\delta H}{\delta q} = r\frac{\delta H}{\delta v_2} - \left(\frac{\delta H}{\delta \phi}\right)' \qquad \frac{\delta H}{\delta r} = q\frac{\delta H}{\delta v_2} + \frac{\delta H}{\delta \psi}. \tag{A.2}$$

Let

$$\frac{\delta H}{\delta K_{(1,2)}} = \partial^{-1}\frac{\delta H}{\delta v_1} + \frac{\delta H}{\delta v_2} + A \tag{A.3}$$

where $A = (A)_{\geqslant 0}$. Then from

$$\delta H = \int \mathrm{res}\left(\frac{\delta H}{\delta K_{(1,2)}}\delta K_{(1,2)}\right) = \int \left(\frac{\delta H}{\delta v_1}\delta v_1 + \frac{\delta H}{\delta v_2}\delta v_2 + \frac{\delta H}{\delta q}\delta q + \frac{\delta H}{\delta r}\delta r\right) \tag{A.4}$$

we have

$$(A)_0 = 0 \tag{A.5}$$

$$(Aq)_0 = \frac{\delta H}{\delta r} - q\frac{\delta H}{\delta v_2} = \frac{\delta H}{\delta \psi} \tag{A.6}$$

$$(A^*r)_0 = \frac{\delta H}{\delta q} - r\frac{\delta H}{\delta v_2} = -\left(\frac{\delta H}{\delta \phi}\right)'. \tag{A.7}$$

Thus A, in fact, is a pure differential operator. Now from (A.1) and (A.2)

$$\begin{aligned}
\frac{\delta H}{\delta K_{(1,2)}}\partial^{-1} &= \left(\partial^{-1}\frac{\delta H}{\delta v_1} + \frac{\delta H}{\delta v_2} + A\right)\partial^{-1} \\
&= \left(\partial^{-1}\frac{\delta H}{\delta v_2} + \partial^{-2}\left(\frac{\delta H}{\delta v_1} + \left(\frac{\delta H}{\delta v_2}\right)'\right) + A\partial^{-1}\right) + \mathrm{O}(\partial^{-3}) \\
&= \left(\partial^{-1}\frac{\delta H}{\delta u_2} + \partial^{-2}\frac{\delta H}{\delta u_1} + A\partial^{-1}\right) + \mathrm{O}(\partial^{-3}) \\
&= \left(\frac{\delta H}{\delta L_{(2,1)}}\right)_- + A\partial^{-1} + \mathrm{O}(\partial^{-3}).
\end{aligned} \tag{A.8}$$

Let us define $B = A\partial^{-1}$, then

$$(B\phi)_0 = (A\partial^{-1}\phi)_0 = (Aq)_0 = \frac{\delta H}{\delta \psi}. \tag{A.9}$$

On the other hand,

$$(B^*\psi)_0 = -(\partial^{-1}A^*r)_0 = -\int^x (A^*r)_0 = \frac{\delta H}{\delta \phi}. \tag{A.10}$$

Equations (A.9) and (A.10) imply that

$$B = \left(\frac{\delta H}{\delta L_{(2,1)}}\right)_+ \tag{A.11}$$

and hence

$$\frac{\delta H}{\delta K_{(1,2)}}\partial^{-1} = \frac{\delta H}{\delta L_{(2,1)}} + \mathrm{O}(\partial^{-3}). \tag{A.12}$$

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